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Gauge invariance on interaction $U(1)$ bundles

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Abstract. The structure of the algebra of gauge-invariant differential forms on the bundle $C \times_M E$ is determined, where $p : C \to M$ is the bundle of connections of a $U(1)$ principal bundle $\pi: P \to M$, and $E \to M$ is the associated bundle to P by the representation λ_r, $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$.

1. Introduction

The aim of this paper is to describe the geometric structure underlying the interaction bundle (i.e. the bundle for interacting particle and gauge fields) in the particular case of $U(1)$ principal bundles. As is well known, the bundle of connections of an arbitrary principal G-bundle $\pi : P \to M$ is an affine bundle $p : C = C(P) \to M$ modelled over the vector bundle $T^*M \otimes \mathrm{ad}P \to M$ (cf [2, 5, 6]), where $\mathrm{ad}P \to M$ is the adjoint bundle: i.e. the bundle associated to P by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. In the particular case $G = U(1)$, which corresponds to classical electromagnetism, the adjoint bundle is canonically isomorphic to the trivial line bundle so that C is an affine bundle modelled over the cotangent bundle T^*M. In two previous papers [7, 8] we proved that, in this case, C is endowed with a canonical symplectic form ω_2 that generates over $\Omega^\bullet(M)$ the algebra of differential forms on C which are invariant under the natural representation of the gauge algebra of P (that is, $\mathrm{gau}P = \Gamma(M, \mathrm{ad}P)$) on C. The initial motivation for that result was the geometric formulation of Utiyama's theorem (cf [3, 4, 6, 14]). If $E \to M$ is the vector bundle associated to P by a linear representation of G on a finite-dimensional real vector space V, the fibred product $C \times_M E$ is usually called the interaction bundle since the Lagrangian for a particle field interacting with a gauge field is defined on it. In fact, this is the bundle on which Utiyama's foundational paper [14] is based (also see [13]), so that it is natural to extend the results of [8] to the interaction bundle in analysing the geometric structure of gauge forms. Moreover, in dealing with the Abelian case we confine ourselves to the linear representations λ_r, $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$, as they are the inequivalent irreducible real representations of $U(1)$ (e.g., see [1, 3.78]).

$\mathcal{I}_{\mathrm{gau}}(C)$ (resp. $\mathcal{I}_{\mathrm{gau}}(E)$, resp. $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$) denotes the algebra of gauge-invariant differential forms on C (resp. E, resp. $C \times_M E$). We have two homomorphisms of $C^\infty(M)$-algebras, $\mathcal{I}_{\mathrm{gau}}(C) \to \mathcal{I}_{\mathrm{gau}}(C \times_M E)$, $\mathcal{I}_{\mathrm{gau}}(E) \to \mathcal{I}_{\mathrm{gau}}(C \times_M E)$. The most outstanding novelty is that $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$ is not generated by $\mathcal{I}_{\mathrm{gau}}(C)$ and $\mathcal{I}_{\mathrm{gau}}(E)$: roughly speaking, in order to generate all gauge-invariant differential forms on the interaction bundle it is necessary to add to the above forms a specific 1-form $\alpha \in \Omega^1(C \times_M E)$, called the interaction 1-form, which depends on the integer r (i.e. on the charge of the particle) and its exterior differential. Accordingly, this form allows one to distinguish different representations by means of the algebra of gauge-invariant forms.

In section 2 we recall some properties of the bundle of connections of a principal bundle and introduce the standard coordinate systems on the bundles C, E, which we use throughout. In section 3 we define the action of the group of automorphisms of P on C, on E and on the interaction bundle, and we obtain the corresponding infinitesimal versions of these actions. This leads us to introduce the notions of gau$P$-invariant and aut$P$-invariant differential forms on the interaction bundle in section 4. In section 5 it is proved that $J^1P \times \mathbb{C}$ is a $U(1)$ principal bundle over $C \times_M E$. In section 6 this bundle structure is used in order to give a direct definition of the interaction 1-form α as being the projection onto $C \times_M E$ of a certain differential form $J^1P \times_M \mathbb{C}$, explicitly defined in terms of the structure form on the 1-jet bundle. The geometric interpretation of the form α is closely related to the classification of the Lagrangians on $J^1(C \times_M E)$ which are gauge invariant in the Utiyama sense (cf [3]). More precisely, for every connection Γ on P and every section $\xi \in \Gamma(M, E)$, $(\sigma_\Gamma, \xi)^*\alpha$ coincides with the imaginary part of $\langle \xi, \nabla\xi \rangle$, where $\sigma_\Gamma : M \to C$ is the section induced by Γ (see the notation below), ∇ is the covariant derivative induced by Γ on E, and $\langle , \rangle$ stands for the standard Hermitian structure on E. The physical meaning of α is also relevant: it is shown to be the 'universal' current of the Yang–Mills–Higgs classical action (see [4, ch 5] and section 6.3 below).

Let A be the standard basis of $\mathfrak{g} = \mathfrak{u}(1)$ (see section 2.2 below for the notation), and let $A^* \in \mathfrak{X}(V)$ be the fundamental vector field associated to A under a linear representation. We set $\mathcal{A}(V) = \{\Omega \in \Omega^\bullet(V); i_{A^*}\Omega = 0, i_{A^*}\,\mathrm{d}\Omega = 0\}$. In section 7 we prove that every differential form $\Omega \in \mathcal{A}(V)$ induces a differential form $\Omega_E \in \Omega^\bullet(E)$, which is not only gauge invariant but also invariant under the Lie algebra of all infinitesimal automorphisms of P. In this way, we obtain all aut$P$-invariant differential forms on E and, furthermore, the structure of $\mathcal{I}_{\mathrm{gau}}(E)$ is determined.

Section 8 is devoted to the statement and proof of the characterization of $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$; as a consequence, we also determine the algebra of aut$P$-invariant forms on $C \times_M E$. Finally, in section 9 we obtain the basic relations among the forms which generate the algebra of gauge-invariant forms.

2. Preliminaries and notation

2.1. The bundle of connections of a principal G-bundle

Let $\pi : P \to M$ be a principal G-bundle over an m-dimensional, connected C^∞ manifold M. We set $T_GP = (TP)/G$, and we denote by $[X]$ the orbit of $X \in TP$ in T_GP. The sections of T_GP correspond with the G-invariant vector fields on P, and π-vertical G-invariant vector fields on P can be identified to the sections of the adjoint bundle. We have an exact sequence of vector bundles over M ([2]), $0 \to \mathrm{ad}P \to T_GP \to TM \to 0$. Connections on P correspond with the splittings of this sequence. Hence, connections on P are the sections of an affine bundle $p : C = C(P) \to M$ modelled over $T^*M \otimes \mathrm{ad}P$. We also denote by $\sigma_\Gamma : M \to C$ the section of the bundle of connections induced from Γ.

2.2. Coordinates in C

Let $\pi : P \to M$ be a $U(1)$ principal bundle and let A be the standard basis of $\mathfrak{u}(1)$: that is, the vector field corresponding to the 1-parameter subgroup $\mathbb{R} \to U(1)$, $t \mapsto \exp(\mathrm{i}t)$.

As $U(1)$ is Abelian, the fundamental vector field $A^* \in \mathfrak{X}(P)$ is $U(1)$ invariant. Set $\tilde{A} = [A^*]$. Let $(W; q^1, \ldots, q^m)$, $m = \dim M$, be an open coordinate domain of M on which $\pi^{-1}(W) \simeq W \times U(1)$. We parametrize the points in $U(1)$ as $\exp(\mathrm{i}t)$, $0 \leqslant t \leqslant 2\pi$. Thus, the functions $(q^1 \circ \pi, \ldots, q^m \circ \pi, t)$ are coordinates on P. We usually identify $q^i \circ \pi$ to q^i. A linear map $\ell : T_q M \to (T_G P)_q$ is a section of π_* if and only if scalars $\lambda_1, \ldots, \lambda_m$ exist such that $\ell(\partial/\partial q^i)_q = (\partial/\partial q^i)_q + \lambda_i \tilde{A}_q$, $1 \leqslant i \leqslant m$. We define $p_1, \ldots, p_m$ on $p^{-1}(W)$ by $p_i(\ell) = -\lambda_i$, or equivalently, for every connection Γ, $\sigma_\Gamma(\partial/\partial q^i) = \partial/\partial q^i - (p_i \circ \sigma_\Gamma)\tilde{A}$. Hence, the functions q^i (or more properly $q^i \circ p$) and p_i, $1 \leqslant i \leqslant m$, are coordinates on $p^{-1}(W)$. The bundle of connections of $M \times U(1)$ can be identified with T^*M, and then the functions (q^i, p_i) have their usual meaning (cf [7, 8]).

2.3. Coordinates in E

Let $\pi_E : E \to M$ be the vector bundle associated to the linear representation λ_r defined in the introduction and let $(W; q^i)$ be as in section 2.2. We denote by $[u, w]$ the orbit of the pair $(u, w) \in P \times \mathbb{C}$ in $E = (P \times \mathbb{C})/U(1)$, and let $s_0 : W \to P$ be the section corresponding to the trivialization $\pi^{-1}(W) \simeq W \times U(1)$. We define functions $\boldsymbol{x}$, $\boldsymbol{y}$ on $\pi_E^{-1}(W)$ by setting $e = [s_0(\pi_E e), \boldsymbol{x}(e) + \mathrm{i}\boldsymbol{y}(e)]$ where $e \in \pi_E^{-1}(W)$ and $(q^1, \ldots, q^m, \boldsymbol{x}, \boldsymbol{y})$ are coordinates on $\pi_E^{-1}(W)$.

2.4. Infinitesimal contact transformations

Let $p : N \to M$ be an arbitrary fibred manifold: i.e. p is a surjective submersion. We denote by $p_1 : J^1 N \to M$ the 1-jet bundle of local sections of p. For every section $s : W \to N$ of p defined on an open subset $W \subseteq M$, we denote by $j^1 s : W \to J^1 N$ its jet prolongation. Set $\dim N = m + n$. Every fibred coordinate system (q^i, y^j), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, for the projection p induces a coordinate system (q^i, y^j, y^j_i) on $J^1 N$ by $y^j_i(j^1_x s) = (\partial(y^j \circ s)/\partial q^i)(x)$. A differential 1-form θ on $J^1 N$ is a contact form if $(j^1 s)^* \theta = 0$ for every local section s of p. The set of all contact forms is a differential system $\mathcal{C}$ of rank n locally generated by $\theta^j = \mathrm{d}y^j - y^j_i\, \mathrm{d}q^i$, $1 \leqslant j \leqslant n$. A vector field $X \in \mathfrak{X}(J^1 N)$ is said to be an infinitesimal contact transformation if $L_X \mathcal{C} \subseteq \mathcal{C}$. For every vector field $X \in \mathfrak{X}(N)$ there exists a unique infinitesimal contact transformation $X^{(1)} \in \mathfrak{X}(J^1 N)$ projecting onto X via the projection $p_{10} : J^1 N \to J^0 N = N$, and the mapping $\mathfrak{X}(N) \to \mathfrak{X}(J^1 N)$, $X \mapsto X^{(1)}$, is a Lie algebra monomorphism (e.g., see [12]).

3. The basic liftings

3.1. The action of the group of automorphism on $C \times_M E$

Let us denote by $\mathrm{Aut}P$ the group of automorphisms of P. Every $\Phi \in \mathrm{Aut}P$ induces a unique diffeomorphism $\phi : M \to M$, such that $\pi \circ \Phi = \phi \circ \pi$. The mapping $\Phi \mapsto \phi$ is a group homomorphism whose kernel is the gauge group, $\mathrm{Gau}P$.

Let ω_Γ be the connection form of a connection Γ on P. Given $\Phi \in \mathrm{Aut}P$ we set $\Phi \cdot \Gamma = \Gamma'$, where $(\Phi^{-1})^* \omega_\Gamma = \omega_{\Gamma'}$ (see [10, section 2.6.2]). As $(\omega_{\Gamma'})_{|\pi^{-1}(\phi x)}$ depends only on $(\omega_\Gamma)_{|\pi^{-1}(x)}$, we can define a unique diffeomorphism $\Phi_C : C \to C$ such that for every connection Γ and every $x \in M$, $\Phi_C(\Gamma(x)) = (\Phi \cdot \Gamma)(x)$. We have (1) $p \circ \Phi_C = \phi \circ p$, (2) $(\Phi \circ \Psi)_C = \Phi_C \circ \Psi_C$, $\forall \Phi, \Psi \in \mathrm{Aut}P$.

Similarly, $\mathrm{Aut}P$ acts on E and on $C \times_M E$ (notation of section 2.3) by setting $\Phi_E([u, w]) = [\Phi(u), w]$, $\bar{\Phi}(\Gamma_x, [u, w]) = (\Phi_C(\Gamma_x), \Phi_E([u, w]))$, respectively, for every

$\Phi \in \mathrm{Aut}P$, with $p(\Gamma_x) = \pi(u) = q$. The definition of $\bar{\Phi}$ makes sense since $p(\Phi_C(\Gamma_x)) = \pi_E(\Phi_E([u,w])) = q$.

3.2. The homomorphism $\mathrm{aut}P \to \mathfrak{X}(C \times_M E)$

Let Φ_t be the local flow of a vector field X on P. Then, X is a $U(1)$-invariant vector field if and only if $\Phi_t \in \mathrm{Aut}P$, $\forall t$. Because of this we denote by $\mathrm{aut}P$ the Lie algebra of all $U(1)$-invariant vector fields on P and we think of the elements of $\mathrm{aut}P$ as being the infinitesimal automorphisms of P. We have an identification $\mathrm{aut}P = \Gamma(M, T_G P)$.

Let Φ_t be the local flow of $X \in \mathrm{aut}P$. We denote by $X_C \in \mathfrak{X}(C)$, $X_E \in \mathfrak{X}(E)$, $\bar{X} \in \mathfrak{X}(C \times_M E)$, the infinitesimal generators of the flows $(\Phi_t)_C$, $(\Phi_t)_E$, $\bar{\Phi}_t$ on $C, E, C \times_M E$, respectively, defined in section 3.1. We have Lie algebra homomorphisms

$$\begin{aligned} &\mathrm{aut}P \to \mathfrak{X}(C) && X \mapsto X_C \\ &\mathrm{aut}P \to \mathfrak{X}(E) && X \mapsto X_E \\ &\mathrm{aut}P \to \mathfrak{X}(C \times_M E) && X \mapsto \bar{X}. \end{aligned}$$

If $(W; q^i)$ is as in section 2.2, then it is not difficult to see that a vector field $X \in \mathfrak{X}(\pi^{-1}W)$ is $U(1)$ invariant if and only if there exist functions $f_i, g \in C^\infty(W)$, such that

$$X = f^i(q^1, \ldots, q^m)\frac{\partial}{\partial q^i} + g(q^1, \ldots, q^m)A^* \tag{1}$$

and we have (see [7, 9]):

$$X_C = f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h\right)\frac{\partial}{\partial p_i} \tag{2}$$

$$X_E = f^i \frac{\partial}{\partial q^i} - rg\left(\boldsymbol{y}\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{y}}\right) \tag{3}$$

$$\bar{X} = f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h\right)\frac{\partial}{\partial p_i} - rg\left(\boldsymbol{y}\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{y}}\right). \tag{4}$$

Note that every $X \in \mathrm{aut}P$ is π-projectable and that the projections of X_C, X_E and $\bar{X}$ onto M coincide with that of X. Moreover, the vector field $X_C + X_E \in \mathfrak{X}(C \times E)$ is tangent to the submanifold $C \times_M E$ and we have $\bar{X} = X_C + X_E$.

4. Invariance

Definition 1. *A differential form Ω on C (resp. E, resp. $C \times_M E$) is said to be* $\mathrm{aut}P$ *invariant if for every $X \in \mathrm{aut}P$, we have*

$$L_{X_C}\Omega = 0 (\textit{resp. } L_{X_E}\Omega = 0, \textit{ resp. } L_{\bar{X}}\Omega = 0). \tag{5}$$

We denote by $\mathcal{I}_{\mathrm{aut}}(C)$ (resp. $\mathcal{I}_{\mathrm{aut}}(E)$, resp. $\mathcal{I}_{\mathrm{aut}}(C \times_M E)$) the algebra of $\mathrm{aut}P$*-invariant differential forms on C (resp. E, resp. $C \times_M E$). From the definitions it follows that there are natural inclusions*

$$\mathcal{I}_{\mathrm{aut}}(C) \subset \mathcal{I}_{\mathrm{aut}}(C \times_M E) \qquad \mathcal{I}_{\mathrm{aut}}(E) \subset \mathcal{I}_{\mathrm{aut}}(C \times_M E)$$

induced by the canonical projections $pr_1 : C \times_M E \to C$, $pr_2 : C \times_M E \to E$, respectively.

A differential form Ω on C (resp. E, resp. $C \times_M E$) is said to be gauge invariant if the corresponding equation in (5) holds true for every $X \in \mathrm{gau}P$. We denote by $\mathcal{I}_{\mathrm{gau}}(C)$ (resp. $\mathcal{I}_{\mathrm{gau}}(E)$, resp. $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$) the algebra of $\mathrm{gau}P$*-invariant forms on C (resp. E,* *resp.* $C \times_M E$*). Note that* $\mathcal{I}_{\text{gau}}(C)$*,* $\mathcal{I}_{\text{gau}}(E)$*,* $\mathcal{I}_{\text{gau}}(C \times_M E)$ *are endowed with a structure of algebra over* $\Omega^\bullet(M)$ *via the natural projections. We thus have natural inclusions*

$$\mathcal{I}_{\text{gau}}(C) \subset \mathcal{I}_{\text{gau}}(C \times_M E) \qquad \mathcal{I}_{\text{aut}}(C) \subset \mathcal{I}_{\text{gau}}(C)$$
$$\mathcal{I}_{\text{gau}}(E) \subset \mathcal{I}_{\text{gau}}(C \times_M E) \qquad \mathcal{I}_{\text{aut}}(E) \subset \mathcal{I}_{\text{gau}}(E)$$
$$\mathcal{I}_{\text{aut}}(C \times_M E) \subset \mathcal{I}_{\text{gau}}(C \times_M E).$$

4.1. Structure of $\mathcal{I}_{\text{gau}}(C)$

The algebra of gauge-invariant forms on the bundle of connections of a $U(1)$ principal bundle $\pi : P \to M$ has been characterized in [7, 8]. It turns out that

$$\mathcal{I}_{\text{gau}}(C) = p^*\Omega^\bullet(M)[\omega_2]$$

where ω_2 is a symplectic form on C whose local expression in the system of coordinates defined in section 2.2 is $\omega_2 = \mathrm{d}p_i \wedge \mathrm{d}q^i$: i.e. the general expression of a gauge-invariant r-form on C is $\xi_r = \sum p^*\nu_{r-2s} \wedge (\omega_2)^s$, $\nu_{r-2s} \in \Omega^\bullet(M)$, $s = 0, \ldots, [n/2]$. In particular, it is not difficult to prove that the subalgebra of aut$P$-invariant forms are polynomial expressions of the form ω_2, that is

$$\mathcal{I}_{\text{aut}}(C) = \mathbb{R}[\omega_2].$$

5. The identification $(J^1P \times \mathbb{C})/U(1) \simeq C \times_M E$

5.1. The connection associated to a point in J^1P

Each section $s : W \to P$ of $\pi : P \to M$, defined on an open neigbourhood of $q \in M$, defines 'an element of connection at q': i.e. a point $\Gamma_q \in C_q$, which is determined by giving a retract $\Gamma_q : T_uP \to V_uP$ of the inclusion of the vertical subspace $V_uP \subset T_uP$, $\forall u \in \pi^{-1}(q)$, as follows: $\Gamma_q(X) = X - (R_z)_* s_* \pi_*(X)$, $X \in T_uP$, where $z \in U(1)$ is the unique element such that $u = s(q) \cdot z$. Note that for every $z \in U(1)$, we have $(R_z)_* \circ \Gamma_q = \Gamma_q \circ (R_z)_*$. It is easy to see that Γ_q depends only on $j^1_q s$, so that we can define a map of fibred manifolds over M, $\gamma : J^1P \to C$ by setting $\gamma(j^1_q s) = \Gamma_q$. We say that $\gamma(j^1_q s)$ is the element of connection at the point q associated to the 1-jet $j^1_q s$.

Proposition 2. *Let us consider the induced action of* $U(1)$ *on* J^1P*; i.e.* $j^1_x s \cdot z = j^1_x(R_z \circ s)$ *for* $z \in U(1)$ *and the action of* $U(1)$ *on* $\mathbb{C}$ *defined by the representation* λ_r*. With the same notation as in sections 2.3 and 5.1, let* $\varphi : J^1P \times \mathbb{C} \to C \times_M E$ *be the map of fibred manifolds over* M *given by* $\varphi(j^1_q s, w) = (\gamma(j^1_q s), [s(q), w])$*. Then,* φ *is a surjective submersion whose fibres are the orbits of the action of* $U(1)$ *on* $J^1P \times \mathbb{C}$ *given by* $(j^1_q s, w) \cdot z = (j^1_q s \cdot z, z^{-1} \cdot w)$*. Hence, we have a natural identification* $(J^1P \times \mathbb{C})/U(1) \simeq C \times_M E$*.*

Proof. Let $\pi_{10} : J^1P \to P$ be the canonical projection, $\pi_{10}(j^1_q s) = s(q)$. With the same notation as in sections 2.2 and 2.4, let (q_i, t, t_i), $1 \leqslant i \leqslant m$, be the coordinate system induced on $\pi_{10}^{-1}(\pi^{-1}(W))$ by $(\pi^{-1}(W); q_i, t)$: i.e. $t_i(j^1_q s) = (\partial(t \circ s)/\partial q^i)(q)$. On $p^{-1}(W) \times_W \pi_E^{-1}(W) \subset C \times_M E$, we consider the coordinate system $(q^i, p_i, \boldsymbol{x}, \boldsymbol{y})$ defined in sections 2.2 and 2.3. In these systems, the equations of φ are

$$q^i \circ \varphi = q^i \qquad p_i \circ \varphi = -t_i (1 \leqslant i \leqslant m) \qquad (\boldsymbol{x} + \mathrm{i}\boldsymbol{y}) \circ \varphi = \exp(\mathrm{i}rt)(x + \mathrm{i}y) \tag{6}$$

thus proving that φ is a submersion. In fact, $(p_i \circ \varphi)(j^1_q s, w) = p_i(\gamma(j^1_q s))$, and from the very definition of the coordinates p_i in section 2.2 we have

$$\sigma_{\gamma(j^1_q s)}(\partial/\partial q^i)_q = [\partial/\partial q^i]_q - p_i(\gamma(j^1_q s))\tilde{A}_q.$$

Hence

$$(\partial/\partial q^i)^*_{s(q)} = (\partial/\partial q^i)_{s(q)} - p_i(\gamma(j^1_q s))\, A^*_{s(q)}.$$

Moreover, according to the definition of the connection associated to $j^1_q s$ in section 5.1, $\gamma(j^1_q s)$ is obtained by imposing

$$\begin{aligned} 0 &= \gamma(j^1_q s)\,(\partial/\partial q^i)^*_{s(q)} = (\partial/\partial q^i)^*_{s(q)} - s_*(\partial/\partial q^i)_q \\ &= ((\partial/\partial q^i)_{s(q)} - p_i(\gamma(j^1_q s))A^*_{s(q)}) - ((\partial/\partial q_i)_{s(q)} + (\partial(t\circ s)/\partial q^i)(q)(\partial/\partial t)_{s(q)}) \end{aligned}$$

and thus

$$p_i(\gamma(j^1_q s)) = -(\partial(t\circ s)/\partial q^i)(q) = -t_i(j^1_q s).$$

Similarly, we have

$$\begin{aligned} ((\boldsymbol{x}+\mathrm{i}\boldsymbol{y})\circ\varphi)(j^1_q s, w) &= (\boldsymbol{x}+\mathrm{i}\boldsymbol{y})([s(q), w]) \\ &= (\boldsymbol{x}+\mathrm{i}\boldsymbol{y})[s_0(q)\cdot \exp(\mathrm{i}t\cdot s(q)), w] \\ &= (\boldsymbol{x}+\mathrm{i}\boldsymbol{y})[s_0(q), \exp(\mathrm{i}rt\cdot s(q))w] \\ &= \exp(\mathrm{i}rt\cdot s(q))w \end{aligned}$$

as follows from the very definition of $\boldsymbol{x}, \boldsymbol{y}$ in section 2.3. Given a point $(\Gamma_q, [u, w]) \in C\times_M E$, $q = \pi(u)$, since Γ_q is a retract of $V_u P \subset T_u P$, we have

$$(\Gamma_q)_{|T_u P} = ((\mathrm{d}t)_u - \lambda_i(\mathrm{d}q^i)_u)\otimes(\partial/\partial t)_u.$$

Hence, we can define a point $j^1_q s \in J^1 P$ by imposing $s(q) = u$, $(\partial(t\circ s)/\partial q^i)(q) = \lambda_i$. Accordingly, Γ_q and $\gamma(j^1_q s)$ coincide over $T_u P$, and since Γ_q and $\gamma(j^1_x s)$ commute with the action of G, we can conclude that Γ_q and $\gamma(j^1_q s)$ coincide at each point of the fibre $\pi^{-1}(q)$. Therefore, φ is surjective. Moreover, since $u = s(q)\cdot z = (s(q)\cdot\zeta)\cdot(\zeta^{-1}z)$, for every $\zeta\in U(1)$, from the definition of γ for every $X\in T_u P$, $u\in\pi^{-1}(q)$, we obtain

$$\gamma(j^1_q(R_\zeta\circ s))(X) = X - (R_{\zeta^{-1}z})_*(R_\zeta\circ s)_*(\pi_* X) = X - s_*\pi_* X = \gamma(j^1_q s)(X).$$

Hence

$$\varphi(j^1_q s\cdot\zeta, \zeta^{-1}\cdot w) = (\gamma(j^1_q s\cdot\zeta), [s(x)\cdot\zeta, \zeta^{-1}\cdot w]) = (\gamma(j^1_q s), [s(q), w]) = \varphi(j^1_q s, w).$$

Conversely, assume $\gamma(j^1_q s) = \gamma(j^1_q s')$, $[s(q), w] = [s'(q), w']$. Then there exists $\zeta\in U(1)$ such that $s'(q) = s(q)\cdot\zeta$, $w' = \zeta^{-1}\cdot w$. Hence $\gamma(j^1_q(R_\zeta\circ s)) = \gamma(j^1_q s')$, and since $(R_\zeta\circ s)(q) = s'(q) = u$, from the definition of γ, we obtain

$$(\partial(t\circ s')/\partial q^i)(q) = (\partial(t\circ R_\zeta\circ s)/\partial q^i)(q).$$

Thus, $j^1_q(R_\zeta\circ s) = j^1_q s\cdot\zeta = j^1_q s'$. □

6. The interaction 1-form

6.1. The structure form

As is well known (e.g., see [12]), $J^1 P$ is endowed with a $V(P)$-valued 1-form θ, called the structure form on the 1-jet bundle. For a $U(1)$ bundle $\pi : P\to M$, the vertical bundle $V(P)$ is a trivial line bundle, so that we can think of the structure form as an ordinary (i.e. real-valued) 1-form on $J^1 P$. With the same notation as in section 2.2, let (q^i, t, t_i), $1\leqslant i\leqslant m$, be the coordinate system induced on $\pi_{10}^{-1}(\pi^{-1}(W))$ by $(\pi^{-1}(W); q^i, t)$. Then, the local expression of the structure form is $\theta = \mathrm{d}t - t_i\,\mathrm{d}q^i$.

Proposition 3. *Let $z = x+\mathrm{i}y$ be the complex coordinate on $\mathbb{C}$, and let $\varphi : J^1P\times\mathbb{C} \to C\times_M E$ be the submersion defined in proposition 2. We have*

(i) The 1-form $\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta$ on $J^1P \times \mathbb{C}$, where θ denotes the structure form and Im the imaginary part, is φ-projectable onto $C \times_M E$: that is, there exists a unique 1-form α on $C \times_M E$ such that

$$\varphi^*(\alpha) = \mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta.$$

(ii) Furthermore, α is $\mathrm{aut}P$-invariant. It is called the interaction 1-form on the bundle $C\times_M E$, and its local expression on the coordinate system $(q^i, p_i, \boldsymbol{x}, \boldsymbol{y})$ (cf sections 2.2 and 2.3) is

$$\alpha = \boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - \boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + r(\boldsymbol{x}^2+\boldsymbol{y}^2)p_i\,\mathrm{d}q^i. \tag{7}$$

Proof. (i) With the notations above the local expression of $\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta$ is $x\,\mathrm{d}y - y\,\mathrm{d}x + r(x^2+y^2)(\mathrm{d}t - t_i\,\mathrm{d}q^i)$. Let $A^\bullet \in \mathfrak{X}(J^1P\times\mathbb{C})$ be the fundamental vector field associated to the standard basis $A \in \mathfrak{u}(1)$ under the action of $U(1)$ on $J^1P\times\mathbb{C}$ defined in proposition 2. We have $A^\bullet = \partial/\partial t + r(y\partial/\partial x - x\partial/\partial y)$. Hence:

(a) $i_{A^\bullet}(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta) = 0$,

(b) $i_{A^\bullet}\,\mathrm{d}(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta) = 0$.

From (a), (b) we obtain $L_{A^\bullet}(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta) = 0$, or equivalently,

(c) $(R_{\exp(\mathrm{i}t)})^*(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta) = \mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta, \forall t \in \mathbb{R}$.

Taking into account that $\ker\varphi_* = \langle A^\bullet\rangle$, by virtue of proposition 2, from equation (a) it follows that $(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z)+rz\bar{z}\theta)(X) = 0$, for every φ-vertical tangent vector $X \in T_{(j^1_q s,w)}(J^1P\times\mathbb{C})$. Moreover, from equation (c) we obtain

$$(\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + r\,z\bar{z}\theta)((R_z)_*X) = (\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta)(X).$$

This proves that there exists a unique 1-form α on $C\times_M E$, such that for every $X \in T(J^1P\times\mathbb{C})$, $\alpha(\varphi_*X) = (\mathrm{Im}\,(\bar{z}\,\mathrm{d}z) + rz\bar{z}\theta)(X)$.

(ii) By using the equations of φ in formula (6), the local expression for α in the statement is easily deduced and, as a simple calculation shows, for every $X \in \mathrm{aut}P$, from formula (4) we obtain $L_{\bar{X}}\alpha = 0$. □

6.2. Hermitian structure on E

As λ_r is a unitary representation, E is endowed with a canonical Hermitian structure $\langle,\rangle$: $E\times_M E \to \mathbb{C}$, which is uniquely determined by imposing $\langle[u, w_1], [u, w_2]\rangle = \bar{w}_1 w_2$, for all $u \in P$, $w_1, w_2 \in \mathbb{C}$, where we have used the notation introduced in section 2.3 and $\bar{w}$ stands for the complex conjugate of $w \in \mathbb{C}$.

The geometric interpretation of the interaction 1-form is as follows.

Proposition 4. *With the hypotheses and notation as in sections 2.1, 2.3 and 6.2, for every connection Γ on $\pi : P \to M$, and every section $\xi \in \Gamma(M, E)$, we have*

$$(\sigma_\Gamma, \xi)^*\alpha = \mathrm{Im}\,\langle\xi, \nabla\xi\rangle \tag{8}$$

where ∇ stands for the covariant derivative induced by Γ on E. Conversely, if β is a pr_2-horizontal 1-form on $C\times_M E$, $pr_2 : C\times_M E \to E$ being the projection onto the second factor, which satisfies the same property stated above, then $\beta = \alpha$.

Proof. As is well known (e.g., see [11, section 3.5.2]) to each section $\xi \in \Gamma(M, E)$ we can associate a function $F_\xi : P \to \mathbb{C}$, by imposing for every $u \in P$, $\xi(\pi(u)) = [u, F_\xi(u)]$

(notation as in section 2.3). If $\chi = (\boldsymbol{x} + \mathrm{i}\boldsymbol{y}) \circ \xi$, locally, then on a trivializing open subset $\pi^{-1}(W) \simeq W \times U(1)$ we have

$$F_\xi(q^1, \ldots, q^m; t) = \exp(-\mathrm{i}rt)\chi(q^1, \ldots, q^m)$$

since $F_\xi(u \cdot z) = z^{-1} F_\xi(u)$, for every $u \in P$, $z \in U(1)$. Given a connection Γ on P, and a vector field X of M, then $X^* F_\xi$ is the function corresponding to the section $\nabla_X \xi$ (cf [10, section 3.1.3]). We have $(\partial/\partial q^i)^* = \partial/\partial q^i - (p_i \circ \sigma_\Gamma)(\partial/\partial t)$. Hence

$$\begin{aligned}(\partial/\partial q^i)^* F_\xi &= \exp(-\mathrm{i}rt)\frac{\partial \chi}{\partial q^i} + \mathrm{i}r \exp(-\mathrm{i}rt)(p_i \circ \sigma_\Gamma)\chi \\ &= \exp(-\mathrm{i}rt)\left(\frac{\partial \chi}{\partial q^i} + \mathrm{i}r(p_i \circ \sigma_\Gamma)\chi\right)\end{aligned}$$

and accordingly,

$$(\boldsymbol{x} + \mathrm{i}\boldsymbol{y}) \circ (\nabla_{\partial/\partial q^i}\xi) = \partial\chi/\partial q^i + \mathrm{i}r(p_i \circ \sigma_\Gamma)\chi.$$

Therefore,

$$\langle \xi, \nabla_{\partial/\partial q^i}\xi \rangle = \bar{\chi}(\partial\chi/\partial q^i) + \mathrm{i}r\chi\bar{\chi}(p_i \circ \sigma_\Gamma)$$

and the result follows from the local expression of α (see formula (7) in proposition 3). Moreover, assume that a pr_2-horizontal 1-form β satisfies the same property as the interaction 1-form. Locally, we have $\beta = A\,\mathrm{d}\boldsymbol{x} + B\,\mathrm{d}\boldsymbol{y} + C_j\,\mathrm{d}q^j$. Hence

$$\begin{aligned}&(A \circ (\sigma_\Gamma, \xi))\frac{\partial(\boldsymbol{x} \circ \xi)}{\partial q^j} + (B \circ (\sigma_\Gamma, \xi))\frac{\partial(\boldsymbol{y} \circ \xi)}{\partial q^j} + C_j \circ (\sigma_\Gamma, \xi) \\ &\qquad = (\boldsymbol{x} \circ \xi)\frac{\partial(\boldsymbol{y} \circ \xi)}{\partial q^j} - (\boldsymbol{y} \circ \xi)\frac{\partial(\boldsymbol{x} \circ \xi)}{\partial q^j} + r\chi\bar{\chi}p_j \circ \sigma_\Gamma.\end{aligned}$$

Since $\boldsymbol{x} \circ \xi$, $\boldsymbol{y} \circ \xi$ are arbitrary functions and for a given $q \in M$, $(\sigma_\Gamma(q), \xi(q))$ is an arbitrary point of the interaction bundle we can conclude $A = -\boldsymbol{y}$, $B = \boldsymbol{x}$, $C_j = r(\boldsymbol{x}^2 + \boldsymbol{y}^2)p_j$, thus concluding the proof. □

Corollary 5. *We have* $(\sigma_\Gamma, \xi)^*\,\mathrm{d}\alpha = 2\mathrm{Im}\,\langle \nabla\xi, \nabla\xi \rangle + r\langle \xi, \xi \rangle(\sigma_\Gamma^*\omega_2)$, *where* ω_2 *is the symplectic 2-form defined in section 4.1.*

Proof. Since ∇ is compatible with the Hermitian metric of E, for every $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}((\sigma_\Gamma, \xi)^*\,\mathrm{d}\alpha)(X, Y) &= \mathrm{d}((\sigma_\Gamma, \xi)^*\alpha)(X, Y) = \mathrm{d}(\mathrm{Im}\,\langle \xi, \nabla\xi \rangle)(X, Y) \\ &= X\mathrm{Im}\,\langle \xi, \nabla_Y\xi \rangle - Y\mathrm{Im}\,\langle \xi, \nabla_X\xi \rangle - \mathrm{Im}\,\langle \xi, \nabla_{[X,Y]}\xi \rangle \\ &= \mathrm{Im}\,(X\langle \xi, \nabla_Y\xi \rangle - Y\langle \xi, \nabla_X\xi \rangle - \langle \xi, \nabla_{[X,Y]}\xi \rangle) \\ &= 2\mathrm{Im}\,\langle \nabla_X\xi, \nabla_Y\xi \rangle + \mathrm{Im}\,\langle \xi, R(X, Y)\xi \rangle\end{aligned}$$

where R is the curvature tensor of ∇. Moreover, from the definition of the coordinates p_i given in section 2.2 and the local expression of ω_2 given in section 4.1, it follows that pulling ω_2 back along the section $\sigma_\Gamma : M \to C$ one obtains the curvature form of Γ: that is, $\sigma_\Gamma^*\omega_2 = \mathrm{d}\omega_\Gamma = \Omega_\Gamma$. The result thus follows from the well known fact on the theory of connections according to which R is the image of Ω_Γ with respect to the homomorphism of Lie algebras induced by the representation under consideration: i.e., in our case $(\lambda_r)_* : \mathfrak{u}(1) \to \mathfrak{gl}(2, \mathbb{R})$, $(\lambda_r)_* \circ \Omega_\Gamma = r\mathrm{i}\Omega_\Gamma = R$. □

6.3. Physical meaning of the interaction form

Let us now consider a pseudo-Riemannian metric $\langle,\rangle_M$ on the base manifold M and a Lagrangian function $\mathcal{L} \in C^\infty(J^1(C \times_M E))$. As is well known (see [4, section 5.1]), for every connection Γ on P and every section ξ of E, an ad$P$-valued 1-form on M is defined: the current $J_{\Gamma,\xi}$, which appears in the inhomogeneous part of the Euler–Lagrange equations of $\mathcal{L}$. In particular, let $\mathcal{L}_{\mathrm{YM}}$ be the classical Abelian Yang–Mills–Higgs Lagrangian, that is

$$\mathcal{L}_{\mathrm{YM}} = \tfrac{1}{2}\langle\nabla\xi, \nabla\xi\rangle_{M,E} - \tfrac{1}{2}m^2\langle\xi, \xi\rangle_E - \tfrac{1}{2}\langle\Omega_\Gamma, \Omega_\Gamma\rangle_M$$

where $\langle,\rangle_E$ is the Hermitian pairing in E defined in section 6.2 and $\langle,\rangle_{M,E}$ denotes the pairing induced by $\langle,\rangle_E$ and the metric tensor $\langle,\rangle_M$ on E-valued differential forms of M. The corresponding current is given by (cf [4, section 5.2])

$$J_{\Gamma,\xi} = \frac{1}{2\mathrm{i}}(\langle\xi, \nabla\xi\rangle_E - \overline{\langle\xi, \nabla\xi\rangle}_E).$$

From the geometrical interpretation of the form α (see proposition 4 above) we obtain

$$J_{\Gamma,\xi} = (\sigma_\Gamma, \xi)^*\alpha.$$

In other words, the interaction form can be understood as a 'universal' current of the Yang–Mills–Higgs action in the sense that its pull-back along a section (σ_Γ, ξ) of the interaction bundle provides the corresponding current.

7. The structure of $\mathcal{I}_{\mathrm{gau}}(E)$

Proposition 6. *Let $\pi_E : E \to M$ be the vector bundle associated to a $U(1)$ principal bundle $\pi : P \to M$ by a linear representation $\lambda : U(1) \to GL(V)$. We denote by $\mathcal{A}(V)$ the algebra of differential forms on V such that $i_{A^*}\Omega = 0$, $i_{A^*}\,\mathrm{d}\Omega = 0$, where $A^* \in \mathfrak{X}(V)$ is the fundamental vector field associated to the standard basis $A \in \mathfrak{u}(1)$ under the linear representation λ. We have*

(i) For every $\Omega \in \mathcal{A}(V)$ of degree d, there exists a unique differential d-form Ω_E on E such that for every $X_1, \ldots, X_d \in T_{(u,w)}(P \times V)$,

$$\Omega_E((\pi_V)_*X_1, \ldots, (\pi_V)_*X_d) = \Omega((pr_2)_*X_1, \ldots, (pr_2)_*X_d)$$

where $\pi_V : P \times V \to E = (P \times V)/U(1)$ is the canonical projection and $pr_2 : P \times V \to V$ is the projection onto the second factor.

(ii) Furthermore, Ω_E is $\mathrm{Aut}(P)$ invariant: i.e. for every $\Phi \in \mathrm{Aut}(P)$, $\Phi_E^*\Omega_E = \Omega_E$. Hence we have a homomorphism of $\mathbb{Z}$-graded algebras $\mathcal{A}(V) \to \mathcal{I}_{\mathrm{aut}}(E)$, $\Omega \mapsto \Omega_E$.

Proof. (i) The formula in the statement completely determines Ω_E. Behaving as in the proof of proposition 3(i), in order to prove the existence of Ω_E we only need to check that $pr_2^*\Omega$ is π_V-projectable, which follows from the hypotheses.

(ii) Every $\Phi \in \mathrm{Aut}P$ acts on an arbitrary associated vector bundle by the same formula as in section 3.1: i.e. $\Phi_E([u, w]) = [\Phi(u), w]$, $u \in P$, $w \in V$, and it is easily seen that $\Phi_E \circ \pi_V = \pi_V \circ (\Phi \times 1_V)$. Hence, for every $X_1, \ldots, X_d \in T_{(u,w)}(P \times V)$ we have

$$\begin{aligned}(\Phi_E^*\Omega_E)((\pi_V)_*X_1, \ldots, (\pi_V)_*X_d) &= \Omega((pr_2)_*(\Phi \times 1_V)_*X_1, \ldots, (pr_2)_*(\Phi \times 1_V)_*X_d)\\ &= \Omega((pr_2)_*X_1, \ldots, (pr_2)_*X_d)\\ &= \Omega_E((\pi_V)_*X_1, \ldots, (\pi_V)_*X_d)\end{aligned}$$

since $pr_2 \circ (\Phi \times 1_V) = pr_2$, thus concluding the proof. □

Notation 7. Let $f : \mathbb{C} \to \mathbb{R}$ be the map $f(z) = \bar{z}z$. It is obvious that $f, \mathrm{d}f \in \mathcal{A}(\mathbb{C})$, under the representation λ_r under consideration. Moreover, as a straightforward computation shows, we have

$$\mathcal{A}(\mathbb{C}) = f^*\Omega^\bullet(\mathbb{R})$$

that is, f and $\mathrm{d}f$ are the generators of $\mathcal{A}(\mathbb{C})$. According to proposition 6(ii), we thus have $f_E, \mathrm{d}f_E \in \mathcal{I}_{\mathrm{aut}}(E)$. For the sake of simplicity, we shall write f instead of f_E. Note that f is the square of the norm of the Hermitian structure on E (cf section 6.2): i.e. $f([u, w]) = \langle [u, w], [u, w] \rangle = \bar{w}w$.

Proposition 8. *Assume M is connected and orientable by a volume form v_m. Then, $\mathcal{I}_{\mathrm{gau}}(E)$ is generated over $(\pi_E, f)^*\Omega^\bullet(M \times \mathbb{R})$ by the globally defined forms $(\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - \boldsymbol{y}\,\mathrm{d}\boldsymbol{x}) \wedge \pi_E^* v_m$ and $\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{y} \wedge \pi_E^* v_m$.*

Proof. Every differential s-form Ω_s on E can be written as follows:

$$\Omega_s = h_I\,\mathrm{d}q^I + h_J^x\,\mathrm{d}q^J \wedge \mathrm{d}\boldsymbol{x} + h_J^y\,\mathrm{d}q^J \wedge \mathrm{d}\boldsymbol{y} + h_K^{xy}\,\mathrm{d}q^K \wedge \mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{y}$$

where $h_I, h_J^x, h_J^y, h_K^{xy} \in C^\infty(E)$, and I, J, K are multi-indices $L = (l_1, \ldots, l_u)$ of degree $|L| = u$ equal to $|I| = s$, $|J| = s - 1$, and $|K| = s - 2$, and we set

$$\mathrm{d}q^L = \mathrm{d}q^{l_1} \wedge \cdots \wedge \mathrm{d}q^{l_u}.$$

If $X = gA^*$, then $X_E = -rg(\boldsymbol{y}\partial/\partial\boldsymbol{x} - \boldsymbol{x}\partial/\partial\boldsymbol{y})$ and by imposing the invariance condition $L_{X_E}\Omega_s = 0$ for $g = 1$ we obtain the following system of equations:

$$X_E(h_I) = 0 \qquad X_E(h_J^x) - rh_J^y = 0 \qquad X_E(h_J^y) + rh_J^x = 0 \qquad X_E(h_K^{xy}) = 0.$$

Hence $h_I, h_K^{xy} \in (\pi_E, f)^*C^\infty(M \times \mathbb{R})$ and the second and third equations above yield

$$h_J^x = A_J\boldsymbol{x} + B_J\boldsymbol{y} \qquad h_J^y = A_J\boldsymbol{y} - B_J\boldsymbol{x}$$

for certain functions $A_J, B_J \in (\pi_E, f)^*C^\infty(M \times \mathbb{R})$. Accordingly, we have

$$\Omega_s = h_I\,\mathrm{d}q^I + A_J\,\mathrm{d}q^J \wedge (\boldsymbol{x}\,\mathrm{d}\boldsymbol{x} + \boldsymbol{y}\,\mathrm{d}\boldsymbol{y}) + B_J\,\mathrm{d}q^J \wedge (\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \boldsymbol{x}\,\mathrm{d}\boldsymbol{y})$$
$$+h_K^{xy}\,\mathrm{d}q^K \wedge \mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{y}.$$

By again imposing the invariance condition for an arbitrary coefficient g, we obtain

$$(\boldsymbol{x}^2 + \boldsymbol{y}^2)B_J\,\mathrm{d}q^J \wedge \mathrm{d}g + h_K^{xy}\,\mathrm{d}q^K \wedge \mathrm{d}g \wedge (\boldsymbol{x}\,\mathrm{d}\boldsymbol{x} + \boldsymbol{y}\,\mathrm{d}\boldsymbol{y}) = 0.$$

Therefore, if $|J| < m$, then $B_J = 0$, and if $|K| < m$, then $h_K^{xy} = 0$ and the result follows. □

Corollary 9. *With the same notation as in propositions 6 and 8 we have*

$$\mathcal{I}_{\mathrm{aut}}(E) = f^*\Omega^\bullet(\mathbb{R}) \simeq \mathcal{A}(\mathbb{C}).$$

8. Structure of $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$

Notation 10. Let $\mathcal{K}$ be the subalgebra of $\Omega^\bullet(C \times_M E)$ defined by

$$\mathcal{K} = (\pi_E \circ pr_2, f \circ pr_2)^*\Omega^\bullet(M \times \mathbb{R})$$

with $pr_1 : C \times_M E \to C$, $pr_2 : C \times_M E \to E$ being the canonical projections onto the factors. Roughly speaking, a form ξ belongs to $\mathcal{K}$ if and only if its local expression in a coordinate system on $C \times_M E$, as in sections 2.2 and 2.3, is

$$\xi = h_{i_1 \ldots i_s}\,\mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_s} + g_{j_1 \ldots j_{s-1}}\,\mathrm{d}q^{j_1} \wedge \cdots \wedge \mathrm{d}q^{j_{s-1}} \wedge \mathrm{d}f$$

where

$$h_{i_1\ldots i_s} = h_{i_1\ldots i_s}(q^1, \ldots, q^n, \boldsymbol{x}^2 + \boldsymbol{y}^2) \qquad g_{j_1\ldots j_{s-1}} = g_{j_1\ldots j_{s-1}}(q^1, \ldots, q^n, \boldsymbol{x}^2 + \boldsymbol{y}^2)$$

are differentiable mappings depending on M and the Hermitian norm of E.

This algebra $\mathcal{K}$, together with the contact form α and the symplectic form $pr_1^*\omega_2$, allows us to state the characterization of $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$ more precisely.

Theorem 11. *Let $\pi : P \to M$ be a $U(1)$ principal bundle, let $p : C \to M$ be the bundle of connections of P, and let $\pi_E : E \to M$ be the vector bundle associated to P by the linear representation λ_r, $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$. With the above hypotheses and notation the forms α, $\mathrm{d}\alpha$, ω_2, generate the algebra of gauge-invariant differential forms on the interaction bundle over the algebra $\mathcal{K}$, where α is the interaction 1-form defined in proposition 3 and ω_2 is the symplectic structure on C defined in section 4.1: that is,*

$$\mathcal{I}_{\mathrm{gau}}(C \times_M E) = \mathcal{K}[\alpha, \mathrm{d}\alpha, pr_1^*\omega_2]. \tag{9}$$

Lemma 12. *A differential form Ω on $C \times_M E$ is* aut$P$ *invariant (resp. gauge invariant) if and only if $\varphi^*\Omega$ is* aut$P$ *invariant (resp. gauge invariant) on $J^1P \times \mathbb{C}$. Moreover, $\mathcal{I}_{\mathrm{aut}}(C \times_M E)$ (resp. $\mathcal{I}_{\mathrm{gau}}(C \times_M E)$) is isomorphic to the algebra of* aut$P$*-invariant (resp. gauge-invariant) differential forms Ξ on $J^1P \times \mathbb{C}$ such that:*

(i) $i_{A^\bullet}\Xi = 0$

(ii) $L_{A^\bullet}\Xi = 0$.

Proof of Lemma 12. The first part of the statement follows from the fact that $(X^{(1)}, 0) \in \mathfrak{X}(J^1P \times \mathbb{C})$ is projectable onto $\tilde{X}$ for every $X \in \mathrm{aut}P$, and the second part follows by taking into account that the fibres of φ are connected. □

Lemma 13. *The algebra of gauge-invariant forms on $J^1P \times \mathbb{C}$ is given by*

$$(\pi_1 \times 1_{\mathbb{C}})^*\Omega^\bullet(M \times \mathbb{C})[\theta, \mathrm{d}\theta]$$

that is, every gauge s-form Ξ on $J^1P \times \mathbb{C}$ can be written as

$$\Xi = \Xi_s + \Xi_{s-1} \wedge \mathrm{d}x + \Xi'_{s-1} \wedge \mathrm{d}y + \Xi_{s-2} \wedge \mathrm{d}x \wedge \mathrm{d}y \tag{10}$$

where Ξ_s, Ξ_{s-1}, Ξ'_{s-1}, Ξ_{s-2} are forms of degree s, $s-1$, $s-1$, $s-2$, respectively, on $J^1P \times \mathbb{C}$, which are polynomials in θ, $\mathrm{d}\theta$ whose coefficients are $(\pi_1 \circ pr_1)$-horizontal differential forms, $pr_1 : J^1P \times \mathbb{C} \to J^1P$ being the canonical projection onto the first factor.

Proof of Lemma 13. First, let us study the gauge invariance on J^1P. Taking into account the local expression of the structure form $\theta = \mathrm{d}t - t_i\,\mathrm{d}q^i$ in a coordinate system (q^i, t, t_i) of J^1P, it is easy to see that every s-form Ξ on J^1P can be locally written as

$$\Xi = \sum_{|I|+|J|=s} f_{IJ}\,(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}t_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{j_n} \wedge \theta$$
$$+ \sum_{|K|+|L|=s} h_{KL}(\mathrm{d}q^1)^{k_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}t_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{l_n}$$

with $f_{IJ}, h_{KL} \in C^\infty(J^1P)$, where $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_n)$, $K = (k_1, \ldots, k_n)$, $L = (l_1, \ldots, l_n)$, are Boolean multi-indices: i.e. $I, J, K, L \in \{0, 1\}^n$, and $|I| = i_1 + \cdots + i_n$. Following the notation in section 3.2, if $X = g(\partial/\partial t)$, $g \in C^\infty(M)$, is the expression of a gauge field on P, its lifting to the jet bundle is

$$X^{(1)} = g\frac{\partial}{\partial t} + \frac{\partial g}{\partial q^i}\frac{\partial}{\partial t_i}.$$

If we let $g = 1$, the condition of gauge invariance tells us the following:

$$0 = L_{X^{(1)}}\Xi = \frac{\partial f_{IJ}}{\partial t}(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}q_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{j_n} \wedge \theta + \frac{\partial h_{KL}}{\partial t}(\mathrm{d}q^1)^{k_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}q_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{l_n}.$$

Hence $\partial f_{IJ}/\partial t = 0$, $\partial h_{KL}/\partial t = 0$. For $g = q^a$, $a = 1, \ldots, n$, we obtain $\partial f_{IJ}/\partial q^a = 0$, $\partial h_{KL}/\partial q^a = 0$ and we conclude that $f_{IJ}, h_{KL} \in C^\infty(M)$, $\forall I, J, K, L$. Now, let us consider $g = \frac{1}{2}(q^1)^2$ in the definition of X. The condition of gauge invariance on the fibre $p^{-1}(q_0)$ yields

$$0 = L_{X^{(1)}}\Xi|_{p^{-1}(q_0)} = f_{IJ}(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}q_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{j_n} \wedge \theta + h_{KL}(\mathrm{d}q^1)^{k_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}q_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}t_n)^{l_n}.$$

Hence if J is such that $j_1 = 1$, then $i_1 = 1$, and if $l_1 = 1$ then $k_1 = 1$. In general, by considering an arbitrary index $1 \leqslant a \leqslant n$ and $g = \frac{1}{2}(q^a)^2$ we conclude that $j_a = 1$ implies $i_a = 1$ and, similarly, $l_a = 1$ implies $k_a = 1$. Therefore, Ξ can be rewritten as

$$\Xi = \sum_{|I|+2|J|=s} \tilde{f}_{IJ}(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}q^1 \wedge \mathrm{d}t_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{j_n} \wedge \theta + \sum_{|K|+2|L|=s} \tilde{h}_{KL}(\mathrm{d}cq^1)^{k_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}q^1 \wedge \mathrm{d}t_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{l_n}$$

with $i_u + j_u \leqslant 1$, $k_u + l_u \leqslant 1$ for $u = 1, \ldots, n$.

If we take $g = q^1 \cdot q^a$, $1 < a \leqslant n$, in the definition of X, the gauge-invariance condition now says

$$\begin{aligned} 0 = L_{X^{(1)}}\Xi|_{p^{-1}(q_0)} &= \tilde{f}_{IJ}(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}q^1 \wedge \mathrm{d}q^a)^{j_1} \\ &\wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{j_n} \wedge \theta + \tilde{f}_{IJ}\,(\mathrm{d}q^1)^{i_1} \wedge \cdots \wedge (\mathrm{d}q^n)^{i_n} \wedge (\mathrm{d}q^1 \wedge \mathrm{d}t_1)^{j_1} \\ &\wedge \cdots \wedge (\mathrm{d}q^a \wedge \mathrm{d}q^1)^{j_a} \wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{j_n} \wedge \theta + \tilde{h}_{KL}(\mathrm{d}q^1)^{k_1} \\ &\wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}q^1 \wedge \mathrm{d}q^a)^{l_1} \wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{l_n} + \tilde{h}_{KL}(\mathrm{d}q^1)^{k_1} \\ &\wedge \cdots \wedge (\mathrm{d}q^n)^{k_n} \wedge (\mathrm{d}t_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}q^a \wedge \mathrm{d}q^1)^{l_a} \wedge \cdots \wedge (\mathrm{d}q^n \wedge \mathrm{d}t_n)^{l_n}. \end{aligned}$$

That is, $\tilde{f}_{IJ} - \tilde{f}_{IJ'} = 0$ whenever

$$J = (1, j_2, \ldots, j_{a-1}, 0, j_{a+1,} \ldots, j_n) \qquad J' = (0, j_2, \ldots, j_{a-1}, 1, j_{a+1}, \ldots, j_n)$$

and $\tilde{h}_{KL} - \tilde{h}_{KL'} = 0$ whenever

$$L = (1, l_2, \ldots, l_{a-1}, 0, l_{a+1,} \ldots, l_n) \qquad L' = (0, l_2, \ldots, l_{a-1}, 1, l_{a+1}, \ldots, l_n).$$

Accordingly, if Ξ contains a summand of the form $\omega_{s-2} \wedge \mathrm{d}q^a \wedge \mathrm{d}t_a$, where $a = 1, \ldots, n$ is an arbitrary fixed index, then Ξ contains the summand $\omega_{s-2} \wedge \mathrm{d}q^1 \wedge \mathrm{d}t_1$, and conversely. Recalling that $\mathrm{d}\theta = \mathrm{d}q^i \wedge \mathrm{d}t_i$, we have that Ξ is a polynomial of θ and $\mathrm{d}\theta$: i.e. $\mathcal{I}_{\mathrm{gau}}(J^1P) = \pi_1^*\Omega^\bullet(M)[\theta, \mathrm{d}\theta]$.

Finally, we note that the gauge group $\mathrm{Gau}P$ acts trivially on $\mathbb{C}$: that is, the action on $J^1P \times \mathbb{C}$ is only defined on the jet bundle. Hence, the result follows. □

Proof of Theorem 11. According to the previous lemmas we are led to study the conditions of φ-projectability $i_{A^\bullet}\Xi = 0$, $L_{A^\bullet}\Xi = 0$, of a form

$$\Xi = \Xi_s + \Xi_{s-1} \wedge \mathrm{d}x + \Xi'_{s-1} \wedge \mathrm{d}y + \Xi_{s-2} \wedge \mathrm{d}x \wedge \mathrm{d}y \tag{11}$$

with Ξ_s, Ξ_{s-1}, Ξ'_{s-1}, Ξ_{s-2} as in lemma 13. The vector field $A^\bullet$ is as follows:

$$A^\bullet = \frac{\partial}{\partial t} + r\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

Hence

$$i_{A^\bullet}\Xi = i_{\partial/\partial t}\Xi_s + (i_{\partial/\partial t}\Xi_{s-1})\wedge \mathrm{d}x + (-1)^{s-1} r y \Xi'_{s-1} + (i_{\partial/\partial t}\Xi'_{s-1})\wedge \mathrm{d}y - (-1)^{s-1} r x \Xi'_{s-1}$$
$$+ (i_{\partial/\partial t}\Xi_{s-2})\wedge \mathrm{d}x\wedge \mathrm{d}y + (-1)^s r\,\Xi_{s-2}\wedge (y\,\mathrm{d}y + x\,\mathrm{d}x)$$

vanishes if and only if

$$\begin{aligned} &i_{\partial/\partial t}\Xi_{s-2} = 0\\ &i_{\partial/\partial t}\Xi_{s-1} + (-1)^s r x\,\Xi_{s-2} = 0\\ &i_{\partial/\partial t}\Xi'_{s-1} + (-1)^s r y\,\Xi_{s-2} = 0\\ &i_{\partial/\partial t}\Xi_s + (-1)^{s-1} r(y\,\Xi_{s-1} - x\,\Xi'_{s-1}) = 0. \end{aligned}$$

As $\theta(\partial/\partial t) = 1$, from the first equation above we conclude that Ξ_{s-2} depends only on $\mathrm{d}\theta$. From the last three equations we obtain

$$\begin{aligned} \Xi_{s-1} &= (-1)^{s-1} r x\theta\wedge\Xi_{s-2} + \xi_{s-1}\\ \Xi'_{s-1} &= (-1)^{s-1} r y\theta\wedge\Xi_{s-2} + \xi'_{s-1}\\ \Xi_s &= (-1)^s r\theta\wedge(y\xi_{s-1} - x\xi'_{s-1}) + \xi_s \end{aligned} \tag{12}$$

where ξ_{s-1}, ξ'_{s-1}, ξ_s are polynomials in $\mathrm{d}\theta$ whose coefficients are $(\pi_1 \circ pr_1)$-horizontal forms. Moreover, substituting the expressions above for Ξ_{s-1}, Ξ'_{s-1}, Ξ_s into formula (11) and simplifying it, we have

$$L_{A^\bullet}\Xi = L_{A^\bullet}\xi_s + (-1)^{s-1} r\theta\wedge(r y\xi'_{s-1} + x L_{A^\bullet}\xi'_{s-1} + r x\xi_{s-1} - y L_{A^\bullet}\xi_{s-1})$$
$$-(-1)^s r\theta\wedge L_{A^\bullet}\Xi_{s-2}\wedge(x\,\mathrm{d}x + y\,\mathrm{d}y) + L_{A^\bullet}\xi_{s-1}\wedge \mathrm{d}x$$
$$+L_{A^\bullet}\xi'_{s-1}\wedge \mathrm{d}y + r(\xi_{s-1}\wedge \mathrm{d}y - \xi'_{s-1}\wedge \mathrm{d}x) + L_{A^\bullet}\Xi_{s-2}\wedge \mathrm{d}x\wedge \mathrm{d}y.$$

Hence $L_{A^\bullet}\Xi = 0$ if and only if

$$\begin{aligned} &L_{A^\bullet}\Xi_{s-2} = 0\\ &L_{A^\bullet}\xi_s = 0\\ &L_{A^\bullet}\xi_{s-1} - r\xi'_{s-1} = 0\\ &L_{A^\bullet}\xi'_{s-1} + r\xi_{s-1} = 0. \end{aligned}$$

As $\mathrm{d}\theta$ does not depend on the variable t, the first two equations above tell us that the coefficients of the differential forms Ξ_{s-2}, ξ_s are invariant under rotations around the origin of the plane $\mathbb{C}$: that is, their dependence on x, y is via the mapping $f = x^2 + y^2$. On the other hand, the last two equations can be seen as a system of partial differential equations and it is not difficult to check that this system is completely integrable and its solution is

$$\begin{aligned} \xi_{s-1} &= x\zeta_{s-1} + y\zeta'_{s-1}\\ \xi'_{s-1} &= y\zeta_{s-1} - x\zeta'_{s-1} \end{aligned} \tag{13}$$

ζ_{s-1}, ζ'_{s-1} being polynomic $s-1$ forms on $\mathrm{d}q's$ and $\mathrm{d}\theta$ whose coefficients are functions of $q^1, \ldots, q^n, x^2 + y^2$.

Taking into account formulae (11)–(13), we finally obtain

$$\Xi = \xi_s - \zeta'_{s-1}\wedge(r(y^2 + x^2)\theta - y\wedge \mathrm{d}x + x\,\mathrm{d}y) + \zeta_{s-1}\wedge(x\,\mathrm{d}x + y\,\mathrm{d}y)$$
$$+\Xi_{s-2}\wedge(r x\,\mathrm{d}x\wedge\theta + r y\,\mathrm{d}y\wedge\theta + \mathrm{d}x\wedge \mathrm{d}y)$$

which projects, by virtue of the local expression of the contact form α (cf proposition 3), onto the form of $C \times_M E$,

$$\xi_s - \tfrac{1}{2} r(\boldsymbol{x}^2 + \boldsymbol{y}^2)\Xi_{s-2}\wedge\omega_2 - \zeta'_{s-1}\wedge\alpha + \tfrac{1}{2}\zeta_{s-1}\wedge \mathrm{d}f + \tfrac{1}{2}\Xi_{s-2}\wedge \mathrm{d}\alpha$$

thus concluding the proof. □

Corollary 14. *The algebra of* aut$P$*-invariant forms on* $C \times_M E$ *is given by*

$$\mathcal{I}_{\mathrm{aut}}(C \times_M E) = f^*\Omega^\bullet(\mathbb{R})[\alpha, \mathrm{d}\alpha, pr_1^*\omega_2].$$

Proof. By virtue of propositions 3(ii) and 6(ii), respectively, the form α and the function f are aut$P$ invariant. From theorem 11 and [8, theorem 3.1], the result thus follows. □

9. Concluding remarks

Remark 15. The fundamental relation among α, f and the symplectic form ω_2 on $C \times_M E$ is

$$\mathrm{d}f \wedge \alpha = f(\mathrm{d}\alpha - rf\omega_2).$$

This follows from proposition 6 and formula (7) in proposition 3, taking into account the local expression of $\omega_2 = \mathrm{d}p_i \wedge \mathrm{d}q^i$.

Notation 16. Let Z be the zero section of E: i.e. $Z = f^{-1}(0)$. We set $Z_C = C \times_M Z$, $\mathcal{O} = C \times_M E—Z_C$. It follows that $\mathcal{O}$ is a dense open subset of the interaction bundle. We denote by $(\pi_E \circ pr_2, f \circ pr_2)_{\mathcal{O}} : \mathcal{O} \to M \times \mathbb{R}$ the restriction of $(\pi_E \circ pr_2, f \circ pr_2)$ to $\mathcal{O}$.

Remark 17. From remark 15 it follows that $\mathrm{d}\alpha|_{\mathcal{O}} = (f^{-1}\,\mathrm{d}f \wedge \alpha + rf\omega_2)|_{\mathcal{O}}$. Hence,

$$\mathcal{I}_{\mathrm{gau}}(\mathcal{O}) = \mathcal{K}[\alpha, pr_1^*\omega_2].$$

Remark 18. Also, in $\mathcal{K}[\alpha, \mathrm{d}\alpha, \omega_2]$, we only need to take one factor for $\mathrm{d}\alpha$, since

$$\mathrm{d}\alpha \wedge \mathrm{d}\alpha = r\omega_2 \wedge (rf\omega_2 + 2\,\mathrm{d}f \wedge \alpha)$$

and the factor $\alpha \wedge \mathrm{d}\alpha$ does not appear either since $\alpha \wedge \mathrm{d}\alpha = r\,f\alpha \wedge \omega_2$. For the sake of simplicity we shall usually identify $\Omega^\bullet(C)$ with $pr_1^*\Omega^\bullet(C)$, and $\Omega^\bullet(E)$ with $pr_2^*\Omega^\bullet(E)$. Accordingly, the general expression for a gauge-invariant n-form Ω_n on the interaction bundle is

$$\Omega_n = \sum_{j=0}^{[\frac{n}{2}]} \eta_{n-2j} \wedge (\omega_2)^j + \sum_{j=0}^{[\frac{n-1}{2}]} \eta'_{n-1-2j} \wedge (\omega_2)^j \wedge \alpha + \sum_{j=0}^{[\frac{n-2}{2}]} \eta''_{n-2-2j} \wedge (\omega_2)^j \wedge \mathrm{d}\alpha$$

where $\eta, \eta', \eta'' \in (\pi_E, f)^*\Omega^\bullet(M \times \mathbb{R})$. Also note that for $n > 2m$, $\Omega_n = 0$, necessarily.

Remark 19. On $\mathcal{O}$, a proof of corollary 5 can also be given by using the formula of remark 15. In fact, if ξ is a non-vanishing section of E on an open subset $U \subset M$, on U we can define an ordinary 1-form by setting $\nabla_X \xi = \eta(X)\xi$, and taking into account that $\xi^*(\mathrm{d}f) = \mathrm{d}\langle\xi, \xi\rangle$ we have

$$\begin{aligned}
(\sigma_\Gamma, \xi)^*(\mathrm{d}f \wedge \alpha)(X, Y) &= ((\mathrm{d}\langle\xi, \xi\rangle) \wedge (\mathrm{Im}\,\langle\xi, \nabla\xi\rangle))(X, Y) \\
&= X\langle\xi, \xi\rangle \cdot \mathrm{Im}\,\langle\xi, \nabla_Y\xi\rangle - Y\langle\xi, \xi\rangle \cdot \mathrm{Im}\,\langle\xi, \nabla_X\xi\rangle \\
&= \mathrm{Im}\,(X\langle\xi, \xi\rangle \cdot \mathrm{Im}\,\langle\xi, \nabla_Y\xi\rangle - Y\langle\xi, \xi\rangle \cdot \langle\xi, \nabla_X\xi\rangle) \\
&= \mathrm{Im}\,(\langle\nabla_X\xi, \xi\rangle\langle\xi, \nabla_Y\xi\rangle - \langle\nabla_Y\xi, \xi\rangle\langle\xi, \nabla_X\xi\rangle) \\
&= \mathrm{Im}\,((\overline{\eta(X)}\eta(Y) - \eta(X)\overline{\eta(Y)})\langle\xi, \xi\rangle^2) \\
&= 2\mathrm{Im}\,(\overline{\eta(X)}\eta(Y))\langle\xi, \xi\rangle^2 \\
&= 2(\mathrm{Im}\,\langle\nabla_X\xi, \nabla_Y\xi\rangle)\langle\xi, \xi\rangle \\
&= \langle\xi, \xi\rangle[((\sigma_\Gamma, \xi)^*\mathrm{d}\alpha)(X, Y) - r\langle\xi, \xi\rangle(\sigma_\Gamma^*\omega_2)(X, Y)].
\end{aligned}$$

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